RELIABILITY ANALYSIS OF AN $N$-COMPONENT SERIES SYSTEM WITH $M$ FAILURE MODES AND VACATION

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Abstract. This paper investigates the reliability characteristics of an $n$-dissimilar-component series repairable system with multiple vacations. Each component has $m$ failure modes with constant failure rates and arbitrary repair time distributions, and the vacation time of the repairman is arbitrary. By using the vector Markov process theory, the supplementary variable method and Laplace transform method, we obtain the explicit expressions of the steady-state availability, the steady-state failure frequency, the steady-state probability that the repairman is on vacation, and the steady-state probability that the system is waiting for repair. In addition, the profit of the system is considered.

Keywords: Reliability, Multiple vacations, Failure modes, Availability, Failure frequency, Profit

1. Introduction. The study of repairable systems is an important topic in reliability. Repairman is one of the essential parts of a repairable system, and can affect the economic benefit of the system, directly or indirectly. Therefore, it has important action on improving the reliability and the benefit by studying the work forms of repairmen in repairable systems. The repairmen leave for a vacation or do other work whenever there are no failed components waiting for repair in repairable systems, which can have important influence to reliability characteristics and economic benefit of repairable systems. Moreover, components consisting of system have a variety of failure modes, and different failure modes can produce different influence. So it is necessary to carry out different measures of repair and replacement.

In early work in this field, some systems with multi-state components have been studied in [1-3]. Most of the work [4,5] deals with only some systems with multiple failure modes, rather than some repairable systems with vacation and multiple failure modes. Past work may be divided into two parts according to the system is studied from the viewpoint of the multiple failure modes or the vacation theory. In the first category we review previous work which relates to multiple failure modes only. An irreparable system with multiple failure modes was first considered by Moore and Shannon [6]. A repairable system with $n$ failure modes and $k$ standby units had carried out by Yamashiro [7]. Song, Liu and Feng [8] dealt with reliability of consecutive $k$-out-of-$n$: $F$ repairable system with $m$ failure modes. In the second category, work is related to a vacation only. The reliability of an $n$-unit series system with multiple vacations of a repairman was considered by Su and Shi [9]. Tang and Liu [10] dealt with the reliability of one unit repairable system with repairman vacation.

For multi-state systems with multi-state components, research efforts have largely been focused on modeling and analysis of reliability. In this paper, we consider an $n$-component series repairable system with $m$ failure modes and multiple vacations. The problem considered in this paper is more general than the work of Su and Shi [9]. The purpose of this
paper is to accomplish two objectives. The first one is to derive the explicit expressions for some steady-state reliability characteristics of the system. The second one is to discuss the profit of the system.

2. Model and Assumptions. The following assumptions are associated with the model:

(1) The system consists of \( n \) dissimilar components and a repairman, each component has \( m \) failure modes. System is operating if and only if all components are working.

(2) When the system fails, the components which are in working order will be temporarily halted and the uptime of the components will be accumulated after the system re-operates.

(3) The failure time distributions are exponential, and the repair time distributions are arbitrary. \( \lambda_{ij} \) and \( \eta_{ij}(y) \) are the failure and repair rates of the component \( i \) with failure mode \( j \). \( (i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m) \). We denote the probability density function and distribution function by \( g_{ij}(y) \) and \( G_{ij}(y) \). The following relationship is clear:

\[
G_{ij}(y) = \int_0^y g_{ij}(t) dt = 1 - \exp \left\{ - \int_0^y \eta_{ij}(t) dt \right\}
\]

Let \( \frac{1}{\eta_{ij}} \) denote the mean time to repair component \( i \) with failure mode \( j \):

\[
\frac{1}{\eta_{ij}} = \int_0^\infty ydG_{ij}(y), \quad i = 1, 2, \ldots, n; j = 1, 2, \ldots, m
\]

(4) If the system is good, the repairman leaves immediately for a vacation. The vacation time has an arbitrary distribution, let \( h(x) \), \( v(x) \) and \( \frac{1}{v} \) denote the probability density function, vacation rate and the mean vacation time, respectively, and then distribution function is denoted by \( H(x) \):

\[
H(x) = \int_0^x h(t) dt = 1 - \exp \left\{ - \int_0^x v(t) dt \right\}, \quad \frac{1}{v} = \int_0^\infty xdH(x)
\]

(5) A repaired component is as good as a new one. All the random variables are independent. Initially, the system with \( n \) new components begins to operate and the repairman begins to leave for a vacation.

3. Model Analysis. In order to describe the different states, we introduce a stochastic process \( \{S(t), t \geq 0\} \) with state space:

\[
J = \{0, 1_{ij}, 2_{ij}; i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}
\]

classified by the following:

- 0: the system is in operation and the repairman is on vacation;
- 1_{ij}: the component \( i \) with failure mode \( j \) is waiting for repair and the repairman is on vacation;
- 2_{ij}: the repairman is repairing the component \( i \) with failure mode \( j \).

Since there are still some general random variables involved, \( \{S(t), t \geq 0\} \) is not a Markov process. For the repairman and each component \( i \) with failure mode \( j \), by introducing the elapsed vacation time \( X(t) \) and the elapsed repair time \( Y_{ij}(t) \) at time \( t \), we can show that the process

\[
\{S(t), X(t), Y_{ij}(t); i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; t \geq 0\}
\]
forms a Markov process with state space:

\[
J^* = \{(0, x), (1_{ij}, x), (2_{ij}, y); 0 \leq x, y < \infty\}
\]

where \( x \) and \( y \) are the value taken by \( X(t) \) and \( Y_{ij}(t) \) respectively.
Now, we define the following state probabilities:

\[
P_0(t, x)\, dx \quad P\{S(t) = 0, x \leq X(t) < x + dx\}
\]

\[
P_{1ij}(t, x)\, dx \quad P\{S(t) = 1_{ij}, x \leq X(t) < x + dx\}
\]

\[
P_{2ij}(t, y)\, dy \quad P\{S(t) = 2_{ij}, y \leq Y_{ij}(t) < y + dy\}
\]

(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)

Throughout this paper we write:

\[
\overline{F}(\cdot) = 1 - F(\cdot), \quad P^* (s) = \int_0^\infty e^{-st} P(t) dt, \quad \Lambda = \sum_{i=1}^n \sum_{j=1}^m \lambda_{ij}
\]

Viewing the nature of this system, the following set of differential equations can be easily set up:

\[
\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \Lambda + v(x) \right] P_0(t, x) = 0 \quad (1)
\]

\[
\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + v(x) \right] P_{1ij}(t, x) = \lambda_{ij} P_0(t, x) \quad (2)
\]

\[
\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \eta_{ij}(y) \right] P_{2ij}(t, y) = 0 \quad (3)
\]

with boundary conditions:

\[
P_0(t, 0) = \int_0^\infty v(x) P_0(t, x) dx + \sum_{i=1}^n \sum_{j=1}^m \int_0^\infty \eta_{ij}(y) P_{2ij}(t, y) dy + \delta(t) \quad (4)
\]

\[
P_{1ij}(t, 0) = 0 \quad (5)
\]

\[
P_{2ij}(t, 0) = \int_0^\infty v(x) P_{1ij}(t, x) dx \quad (6)
\]

and initial conditions:

\[
P_0(0, x) = \delta(x), \quad P_{1ij}(0, x) = 0, \quad P_{2ij}(0, y) = 0, \quad (i = 1, 2, \cdots n, j = 1, 2, \ldots, m).
\]

where \(\delta(x)\) is the Dirac delta function.

Taking the Laplace transform of the equations (1)-(6), as well as initial conditions, we have:

\[
P_0^*(s, x) = \overline{F}(x)e^{-(s+\Lambda)x}C_0(s) \quad (7)
\]

\[
P_{1ij}^*(s, x) = \frac{\lambda_{ij}}{\Lambda} \overline{F}(x)[e^{-sx} - e^{-(s+\Lambda)x}]C_0(s) \quad (8)
\]

\[
P_{2ij}^*(s, y) = \frac{\lambda_{ij}}{\Lambda} \overline{G}_{ij}(y)[h^*(s) - h^*(s + \Lambda)]C_0(s) \quad (9)
\]

where

\[
C_0(s) = \left\{ 1 - h^*(s + \Lambda) - [h^*(s) - h^*(s + \Lambda)] \sum_{i=1}^n \sum_{j=1}^m \frac{\lambda_{ij}}{\Lambda} g_{ij}^*(s) \right\}^{-1}
\]

4. **Reliability Characteristics.** According to the probability analysis of the system in Section 3, we can obtain the reliability characteristics of the system as follows.
4.1. **Availability of the system.** The availability of the system, denoted by \( A(t) \), is the probability that the system is operating at time \( t \).

**Theorem 4.1.** The Laplace transform of \( A(t) \) is

\[
A^*(s) = \mathcal{H}^*(s + \Lambda) \left\{ 1 - h^*(s + \Lambda) - [h^*(s) - h^*(s + \Lambda)] \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\lambda_{ij}}{\Lambda} g_{ij}^*(s) \right\}^{-1}
\]

and the steady-state availability of the system, denoted by \( A \), is

\[
A = \mathcal{H}^*(\Lambda) \left[ \frac{1}{v} + \mathcal{H}^*(\Lambda) \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} \eta_{ij} \right]^{-1}
\]

**Proof:** Based on definition of the availability of the system and the fact that the system is operating if and only if the stochastic process \( S(t) \) is in state 0, we know:

\[
A(t) = \int_{0}^{\infty} P_0(t, x) dx
\]

Taking the Laplace transform of the above equation, we get

\[
A^*(s) = \int_{0}^{\infty} P_0^*(s, x) dx
\]

Substituting (7) into the above equation, we obtain (10). By the terminal-value theorem of the Laplace transform, we have

\[
A = \lim_{t \to \infty} A(t) = \lim_{s \to 0} s A^*(s).
\]

4.2. **Failure frequency of the system.** The failure frequency of the system, denoted by \( m_f(t) \), is the rate of occurrence of failures of the system during \((0, t]\).

**Theorem 4.2.** The Laplace transform of \( m_f(t) \) is given by

\[
m_f^*(s) = \Lambda \mathcal{H}^*(s + \Lambda) \left\{ 1 - h^*(s + \Lambda) - [h^*(s) - h^*(s + \Lambda)] \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\lambda_{ij}}{\Lambda} g_{ij}^*(s) \right\}^{-1}
\]

and the steady-state failure frequency of the system, denoted by \( m_f \), is

\[
m_f = \Lambda \mathcal{H}^*(\Lambda) \left[ \frac{1}{v} + \mathcal{H}^*(\Lambda) \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} \eta_{ij} \right]^{-1}
\]

**Proof:** Based on the definition of \( m_f(t) \) and the method in Ref. [11], we know that:

\[
m_f(t) = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{0}^{\infty} \lambda_{ij} P_0(t, x) dx = \Lambda \int_{0}^{\infty} P_0(t, x) dx
\]

Taking the Laplace transform of the above equation, we have

\[
m_f^*(s) = \Lambda \int_{0}^{\infty} P_0^*(s, x) dx
\]

Substituting (7) into the above equation, we obtain (12). By the terminal-value theorem of the Laplace transform, we have

\[
m_f = \lim_{t \to \infty} m_f(t) = \lim_{s \to 0} s m_f^*(s)
\]
4.3. **The probability that the repairman is on vacation.** Let \( P_V(t) \) denote the probability that the repairman is on vacation at time \( t \).

**Theorem 4.3.** The Laplace transform of \( P_V(t) \) is

\[
P_V^*(s) = \mathcal{H}^*(s) \left\{ 1 - h^*(s + \Lambda) - [h^*(s) - h^*(s + \Lambda)] \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\lambda_{ij}}{\Lambda} g_{ij}^*(s) \right\}^{-1}
\]

(14)

and the steady-state probability that the repairman is on vacation, denoted by \( P_V \), is

\[
P_V = \frac{1}{v} \left[ \frac{1}{v} + \mathcal{H}^*(\Lambda) \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} \eta_{ij} \right]^{-1}
\]

(15)

**Proof:** According to the assumptions of the system, we know:

\[
P_V(t) = \int_0^\infty P_0(t, x) dx + \sum_{i=1}^{n} \sum_{j=1}^{m} \int_0^\infty P_{1_{ij}}(t, x) dx
\]

Taking the Laplace transform of the above equation, we have

\[
P_V^*(s) = \int_0^\infty P_0^*(s, x) dx + \sum_{i=1}^{n} \sum_{j=1}^{m} \int_0^\infty P_{1_{ij}}^*(s, x) dx
\]

Substituting (7) and (8) into the above equation, we obtain (14). By the terminal-value theorem of the Laplace transform, we have

\[
P_V = \lim_{t \to \infty} P_V(t) = \lim_{s \to 0} sP_V^*(s).
\]

4.4. **The probability that the system is waiting for repair.** Let \( P_W(t) \) denote the probability that the system is waiting for repair at time \( t \).

**Theorem 4.4.** The Laplace transform of \( P_W(t) \) is

\[
P_W^*(s) = \left[ \mathcal{H}^*(s) - \mathcal{H}^*(s + \Lambda) \right] \left\{ 1 - h^*(s + \Lambda) - [h^*(s) - h^*(s + \Lambda)] \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\lambda_{ij}}{\Lambda} g_{ij}^*(s) \right\}^{-1}
\]

and the steady-state probability that the system is waiting for repair, denoted by \( P_W \), is

\[
P_W = \left[ \frac{1}{v} - \mathcal{H}^*(\Lambda) \right] \left[ \frac{1}{v} + \mathcal{H}^*(\Lambda) \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} \eta_{ij} \right]^{-1}
\]

(17)

**Proof:** According to the assumptions of the system, we know:

\[
P_W(t) = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_0^\infty P_{1_{ij}}(t, x) dx
\]

Taking the Laplace transform of the above equation, we have

\[
P_W^*(s) = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_0^\infty P_{1_{ij}}^*(s, x) dx
\]

Substituting (8) into the above equation, we obtain (16). By the terminal-value theorem of the Laplace transform, we have

\[
P_W = \lim_{t \to \infty} P_W(t) = \lim_{s \to 0} sP_W^*(s).
\]
5. **Profit of the System.** We develop an expected profit function per unit time for the system. Our objective is to determine that the repairman leaves for a vacation or dose not.

Let

\( x_1 \equiv \text{revenue per unit time when the system is operating}, \)

\( x_2 \equiv \text{revenue per unit time when the repairman is on vacation}, \)

\( x_3 \equiv \text{loss when the system breaks down for each time}. \)

Then the profit per unit time of the system with repairman vacation, \( P(c, \lambda_{ij}, \eta_{ij}) \), is given by

\[
P(c, \lambda_{ij}, \eta_{ij}) = Ax_1 + P_v x_2 - m_f x_3, \quad (i = 1, \ldots, n, j = 1, \ldots, m).
\]

The profit per unit time of the system without repairman vacation, \( P(\lambda_{ij}, \eta_{ij}) \), is given by

\[
P(\lambda_{ij}, \eta_{ij}) = A_0 x_1 - m_f x_3, \quad (i = 1, \ldots, n, j = 1, \ldots, m)
\]

where

\[
A_0 = \left[ 1 + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\lambda_{ij}}{\eta_{ij}} \right]^{-1}, \quad m_f = \Lambda \left[ 1 + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\lambda_{ij}}{\eta_{ij}} \right]^{-1}
\]

Then the expected profit per unit time, \( B(c, \lambda_{ij}, \eta_{ij}) \), is given by

\[
B(c, \lambda_{ij}, \eta_{ij}) = P(c, \lambda_{ij}, \eta_{ij}) - P(\lambda_{ij}, \eta_{ij}) = (A - A_0) x_1 + P_v x_2 + (m_f - m_f) x_3
\]

Thus when \( B(c, \lambda_{ij}, \eta_{ij}) > 0 \), the repairman leaves for vacation; when \( B(c, \lambda_{ij}, \eta_{ij}) < 0 \), the repairman does not.

6. **Conclusions.** In this paper, we demonstrated that the vector Markov process theory and the supplementary variable method work efficiently for the \( n \)-component series repairable system with \( m \) failure modes and multiple vacations. The system discussed can be deemed as an extension of an \( n \)-component series repairable system with repairman vacation (see Su and Shi [9]), which is one of the important repairable systems we often encounter in reliability applications. It is shown that the results are more general than existing results in literature. Moreover, we analyzed the profit of the system.

**REFERENCES**


