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THE PARAMETRIZATION FOR THE CLASS OF ALL PROPER INTERNALLY STABILIZING CONTROLLERS FOR MULTIPLE-INPUT/MULTIPLE-OUTPUT MINIMUM PHASE SYSTEMS

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Abstract. *In the present paper, we examine the parametrization of all proper internally stabilizing controllers for multiple-input/multiple-output minimum phase systems. Glaria and Goodwin first gave the parametrization of all stabilizing controllers for single-input/single-output minimum phase systems. Their parametrization has practical difficulties. One is that the free parameter of their parametrization has constraint for the controller to be proper. The other is that they do not give the parametrization of all internally stabilizing controllers. Yamada overcame these problems and proposed the parametrization of all proper internally stabilizing controllers for single-input/single-output minimum phase systems. However, no paper examines the parametrization of all proper internally stabilizing controllers for multiple-input/multiple-output minimum phase systems. In this paper, we propose the parametrization of all proper internally stabilizing controllers for multiple-input/multiple-output minimum phase systems.*
Keywords: Multiple-input/multiple-output system, Minimum phase system, Parametrization

1. **Introduction.** In the present paper, we examine the parametrization of all proper internally stabilizing controllers for multiple-input/multiple-output minimum phase systems. The parametrization problem is the problem in which all stabilizing controllers for a plant are sought [1, 2, 3, 4, 5, 6, 7, 8]. Since this parametrization can successfully search for all proper stabilizing controllers, it is used as a tool for many control problems.

For a stable plant, the parametrization of all stabilizing controllers has a structure identical to that of Internal Model Control [5]. For an unstable plant, the structure of a parametrization of all stabilizing controllers has full-order state feedback, including a full-order observer [7]. Glaria and Goodwin [6] gave a simple parametrization for single-input/single-output minimum phase systems. However, two difficulties remain. One is that the parametrization of all stabilizing controller given by Glaria and Goodwin generally includes improper controllers. In practical application, the controller is required to be proper. The other is that they do not give the parametrization of all internally stabilizing controllers. Yamada overcame these problems and proposed the parametrization of all proper internally stabilizing controllers for single-input/single-output minimum phase systems [10]. The parametrization of all stabilizing controllers in [8] is applied to many control problems such as the parametrization of all stabilizing modified repetitive controllers for minimum phase systems [9], adaptive control systems [11], model feedback control systems [12], parallel compensation technique [13], PI control [14] and PID control [15]. If the parametrization of all stabilizing controllers for multiple-input/multiple-output minimum phase systems is obtained, the results in [9, 10, 11, 12, 13, 14, 15] are expanded for multiple-input/multiple-output minimum phase systems. However, no paper examines the parametrization of all internally stabilizing controllers for multiple-input/multiple-output minimum phase systems.

In this paper, we expand the result in [8] and give the parametrization of all proper internally stabilizing controllers for the multiple-input/multiple-output continuous time minimum phase systems. In order to guarantee proper controllers, a restriction of the free parameter $Q(s)$ by Glaria and Goodwin is traded by a choice of an adequate rational function $K(s)$. It is shown that there exists an adequate rational function $K(s)$. To achieve the internally stability condition, it is shown that the free parameter $Q(s)$ must be biproper stable rational function. This parametrization is simple and does not contain improper controller, that is, this parametrization contains only proper internally stabilizing controllers.

2. **Problem Formulation.** Consider the unity feedback control system in

$$\left\{\begin{array}{rcl} y & = & G(s)u \\ u & = & -C(s)y \end{array}\right. , \tag{1}$$

where $G(s) \in R^{p\times p}(s)$ is the plant, $C(s) \in R^{p\times p}(s)$ is the controller, $u \in R^p$ is the control input and $y \in R^p$ is the output. $G(s)$ is assumed to be proper and of minimum phase, that is, $G(s)$ has no zero in the closed right half plane. In addition, $G(s)$ is assumed satisfying

$$\text{rank}\, G(s) = p. \tag{2}$$

The problem considered in this paper is to obtain the parametrization of all proper controllers to make the control system in (1) internally stable.

3. **Parametrization for Biproper Plant.** In this section, we propose the parametrization of all proper internally stabilizing controllers for minimum phase biproper plant $G(s)$.

In this section, $G(s)$ is assumed to be biproper, that is,

$$\det \lim_{s\to\infty} \{G(s)\} \neq 0 \tag{3}$$

holds true. The parametrization of all proper internally stabilizing controllers for $G(s)$ satisfying (3) is given by following theorem.

Theorem 3.1. *$G(s)$ is assumed to be of minimum phase and biproper. A proper controller $C(s)$ makes the control system in (1) internally stable if and only if $C(s)$ is written by*

$$C(s) = Q^{-1}(s) - G^{-1}(s), \tag{4}$$

where $Q(s) \in RH_\infty$ is any non-singular and biproper rational function.

Proof: First, the necessity is proved. That is, it is shown that if $C(s)$ makes the control system in (1) internally stable, then $C(s)$ is written by (4). From the assumption that $C(s)$ makes the control system in (1) internally stable, $(I + G(s)C(s))^{-1} \in RH_\infty$, $(I + G(s)C(s))^{-1}G(s) \in RH_\infty$, $C(s)(I+G(s)C(s))^{-1} \in RH_\infty$ and $(I+G(s)C(s))^{-1}G(s)C(s) \in RH_\infty$ hold true. Let

$$Q(s) = (I + G(s)C(s))^{-1} G(s). \tag{5}$$

From (2), $Q(s)$ in (5) is obviously biproper. In addition, from the assumption that $C(s)$ makes the control system in (1) internally stable, $Q(s)$ in (5) is included in RH_∞. From (5), we have (4). Thus, the necessity has been shown.

Next, the sufficiency is shown. That is, we show that if $C(s)$ is written by (4), then the control system in (1) is internally stable. Since $C(s)$ is written by (4), $(I + G(s)C(s))^{-1}$,

$(I+G(s)C(s))^{-1}G(s)$, $C(s)(I+G(s)C(s))^{-1}$ and $(I+G(s)C(s))^{-1}G(s)C(s)$ are written by

$$(I+G(s)C(s))^{-1} = Q(s)G^{-1}(s), \tag{6}$$

$$(I+G(s)C(s))^{-1}G(s) = Q(s), \tag{7}$$

$$C(s)(I+G(s)C(s))^{-1} = \left(Q^{-1}(s) - G^{-1}(s)\right) Q(s)G^{-1}(s) \tag{8}$$

and

$$(I+G(s)C(s))^{-1}G(s)C(s) = I - Q(s)G^{-1}(s), \tag{9}$$

respectively. Since $Q(s) \in RH_\infty$ and $G(s)$ is of minimum phase, all of transfer functions in (6)∼(9) are stable. In addition, from the assumption that both $Q(s)$ and $G(s)$ are biproper, all of transfer functions in (6)∼(9) are proper. Thus, the sufficiency has been shown.

We have thus proved Theorem 3.1.

Note 3.1. *From Theorem 3.1, when $G(s)$ is biproper and of minimum phase, we can obtain the parametrization of all proper internally stabilizing controllers by identifying $G^{-1}(s)$.*

4. **The Parametrization for Strictly Proper Plant.** In this section, we propose the parametrization of all proper internally stabilizing controllers for minimum phase strictly proper plant $G(s)$.

In this section, $G(s)$ is assumed to be strictly proper, that is,

$$\det \lim_{s\to\infty} \{G(s)\} = 0 \tag{10}$$

holds true. The parametrization of all proper internally stabilizing controllers for $G(s)$ satisfying (10) is given by following theorem.

Theorem 4.1. *$G(s)$ is assumed to be of minimum phase and strictly proper. There exists $K(s)$ that satisfies the following expressions:(1)$G(s)+K(s)$ is of minimum phase. (2)$K(s)$ is biproper and stable. Using above mentioned $K(s)$, the parametrization of all proper internally stabilizing controllers $C(s)$ for $G(s)$ is given by*

$$C(s) \;=\; \left(I+\bar{C}(s)K(s)\right)^{-1}\bar{C}(s) \quad \left(\det \lim_{\omega\to\infty}\left(I+\bar{C}(j\omega)K(j\omega)\right) \neq 0\right), \tag{11}$$

where $\bar{C}(s)$ is denoted by

$$\bar{C}(s) = \bar{Q}^{-1}(s) - (G(s)+K(s))^{-1} \tag{12}$$

and $\bar{Q}(s) \in RH_\infty$ is any non-singular and biproper rational function.

The proof of Theorem 4.1 requires the following theorems.

Theorem 4.2. *If $\bar{G}(s) = G(s)+K(s)$ is of minimum phase and biproper, then the parametrization of all proper internally stabilizing controllers $\bar{C}(s)$ for $\bar{G}(s)$ is written by*

$$\bar{C}(s) = \bar{Q}^{-1}(s) - \bar{G}^{-1}(s) = \bar{Q}^{-1}(s) - (G(s)+K(s))^{-1}, \tag{13}$$

where $\bar{Q}(s) \in RH_\infty$ is any non-singular and biproper rational function.

Proof: Since $\bar{G}(s) = G(s)+K(s)$ is assumed to be minimum phase and biproper, from Theorem 3.1, it is obvious that the parametrization of all stabilizing controllers for $\bar{G}(s) = G(s)+K(s)$ is given by (13).

This completes the proof of this theorem.

Theorem 4.3. *If $G(s)$ is of minimum phase and strictly proper, there exists $K(s)$ that satisfies the following expressions:(1)$G(s)+K(s)$ is of minimum phase. (2)$K(s)$ is biproper and stable.*

Proof: $G(s) + K(s)$ is written as

$$G(s) + K(s) = (N(s) + K(s)D(s))\, D^{-1}(s), \tag{14}$$

where $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are right coprime factors of $G(s)$ over RH_∞ written by

$$G(s) = N(s)D^{-1}(s). \tag{15}$$

The existence condition of $K(s) \in RH_\infty$ to satisfy $N(s) + K(s)D(s) \in \mathcal{U}$ is equivalent to that of $U(s) \in \mathcal{U}$ and $K(s) \in RH_\infty$ satisfying

$$U(s) = N(s) + K(s)D(s) \in \mathcal{U}, \tag{16}$$

where $\mathcal{U}$ is the set of unimodular matrices on RH_∞, thus $U(s) \in \mathcal{U}$ means $U(s) \in RH_\infty$ and $U^{-1}(s) \in RH_\infty$. Applying Theorem 4.4.2 in [4] for (16), the existence condition of $U(s)$ and $K(s)$ is reduced to that of $u(s) \in \mathcal{U}$ and $k(s) \in RH_\infty$ satisfying

$$u(s) = n(s) + k(s)d(s) \in \mathcal{U}, \tag{17}$$

where $n(s) = \det(N(s)) \in RH_\infty$ and $d(s) \in RH_\infty$ denotes the smallest invariant factor of $D(s)$. The existence condition of $u(s)$ and $k(s)$ is equivalent to the interpolation problem written by

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}u(s_i) = \frac{\mathrm{d}^j}{\mathrm{d}s^j}n(s_i) \ \ (j = 0, \dots, m_i - 1; i = 1, \dots, l), \tag{18}$$

where $s_1, \dots, s_l$ are distinct zeros of $d(s)$ on the positive real axis and $m_1, \dots, m_l$ are the corresponding multiplicities. Since $G(s)$ is of minimum phase, $n(s)$ is also of minimum phase. This implies that all of $n(s_i)$ are the same sign. From Theorem 2.3.3 in [4], there exists $u(s) \in \mathcal{U}$ and $k(s) \in RH_\infty$ satisfying (18). This implies that there exists $U(s) \in \mathcal{U}$ and $K(s) \in RH_\infty$ satisfying (16).

The remaining problem is whether or not, $K(s)$ is biproper. Next, it is shown that if $U(s) \in \mathcal{U}$ exists such that (16) holds true, then $K(s)$ is biproper. From (16), $K(s)$ is written by

$$K(s) = U(s)D^{-1}(s) - N(s)D^{-1}(s). \tag{19}$$

The assumption that $U(s)$ holds (16) implies that $K(s)$ written by (19) is stable. Since both $U(s)$ and $D(s)$ are biproper and $N(s)$ is strictly proper, $K(s)$ denoted by (19) is biproper.

This completes the proof of this theorem.

Theorem 4.4. *It is assumed that $K(s) \in RH_\infty$ is biproper and $G(s)$ is strictly proper. If $C(s)$ stabilizes $G(s)$, then $\bar{C}(s)$ written by*

$$\bar{C}(s) = (I - C(s)K(s))^{-1} C(s) \ \ \left(\det\left\{\lim_{\omega\to\infty} (I - C(j\omega)K(j\omega))\right\} \neq 0\right) \tag{20}$$

stabilizes $\bar{G}(s) = G(s) + K(s)$. In addition, the reverse is also true. That is, if $\bar{C}(s)$ stabilizes $\bar{G}(s) = G(s) + K(s)$, then $C(s)$ written by

$$C(s) = \left(I + \bar{C}(s)K(s)\right)^{-1} \bar{C}(s) \ \ \left(\det\left\{\lim_{\omega\to\infty} \left(I - \bar{C}(j\omega)K(j\omega)\right)\right\} \neq 0\right) \tag{21}$$

stabilizes $G(s)$.

Proof: We will show that if $C(s)$ stabilizes $G(s)$, then $\bar{C}(s)$ written by (20) stabilizes $\bar{G}(s) = G(s) + K(s)$. From (20) and simple manipulation, $(I + \bar{G}(s)\bar{C}(s))^{-1}$, $(I + \bar{G}(s)\bar{C}(s))^{-1}\bar{G}(s)$, $\bar{C}(s)(I + \bar{G}(s)\bar{C}(s))^{-1}$ and $(I + \bar{G}(s)\bar{C}(s))^{-1}\bar{G}(s)\bar{C}(s)$ are rewritten by

$$\left(I + \bar{G}(s)\bar{C}(s)\right)^{-1} = (I - K(s)C(s))\,(I + G(s)C(s))^{-1}, \tag{22}$$

$$\left(I + \bar{G}(s)\bar{C}(s)\right)^{-1}\bar{G}(s) = (I - K(s)C(s))\,(I + G(s)C(s))^{-1}\,(G(s) + K(s)), \tag{23}$$

$$\bar{C}(s)\left(I+\bar{G}(s)\bar{C}(s)\right)^{-1}=C(s)\left(I+G(s)C(s)\right)^{-1} \tag{24}$$

and

$$\left(I+\bar{G}(s)\bar{C}(s)\right)^{-1}\bar{G}(s)\bar{C}(s)=\left(G(s)+K(s)\right)C(s)\left(I+G(s)C(s)\right)^{-1}, \tag{25}$$

respectively. From the assumption that $C(s)$ stabilizes $G(s)$, $(I+G(s)C(s))^{-1}\in RH_\infty$, $(I+G(s)C(s))^{-1}G(s)\in RH_\infty$, $C(s)(I+G(s)C(s))^{-1}\in RH_\infty$ and $(I+G(s)C(s))^{-1}G(s)C(s)\in RH_\infty$ hold true. Therefore, all of transfer functions in (22)$\sim$(25) are included in RH_∞.

Next, we will show that if $\bar{C}(s)$ stabilizes $\bar{G}(s)=G(s)+K(s)$, then $C(s)$ written by (21) stabilizes $G(s)$. From (21) and simple manipulation, $(I+G(s)\bar{C}(s))^{-1}$, $(I+G(s)C(s))^{-1}G(s)$, $C(s)(I+G(s)C(s))^{-1}$ and $(I+G(s)C(s))^{-1}G(s)C(s)$ are rewritten by

$$\left(I+G(s)C(s)\right)^{-1}=\left(I+K(s)\bar{C}(s)\right)\left(I+\bar{G}(s)\bar{C}(s)\right)^{-1}, \tag{26}$$

$$\left(I+G(s)C(s)\right)^{-1}G(s)=\left(I+K(s)\bar{C}(s)\right)\left(I+\bar{G}(s)\bar{C}(s)\right)^{-1}\left(\bar{G}(s)-K(s)\right), \tag{27}$$

$$C(s)\left(I+G(s)C(s)\right)^{-1}=\bar{C}(s)\left(I+\bar{G}(s)\bar{C}(s)\right)^{-1} \tag{28}$$

and

$$\left(I+G(s)C(s)\right)^{-1}G(s)C(s)=\left(\bar{G}(s)-K(s)\right)\bar{C}(s)\left(I+\bar{G}(s)\bar{C}(s)\right)^{-1}, \tag{29}$$

respectively. From the assumption that $\bar{C}(s)$ stabilizes $\bar{G}(s)$, $(I+\bar{G}(s)\bar{C}(s))^{-1}\in RH_\infty$, $(I+\bar{G}(s)\bar{C}(s))^{-1}\bar{G}(s)\in RH_\infty$, $\bar{C}(s)(I+\bar{G}(s)\bar{C}(s))^{-1}\in RH_\infty$ and $(I+\bar{G}(s)\bar{C}(s))^{-1}\bar{G}(s)\bar{C}(s)\in RH_\infty$ hold true. Therefore, all of transfer functions in (26)$\sim$(29) are included in RH_∞.

We have thus proved Theorem 4.4.

Theorem 4.1 is proved using the above-described theorems. **Proof:** From Theorem 4.3, there exists biproper $K(s)\in RH_\infty$ to make $\bar{G}(s)=G(s)+K(s)$ of minimum phase. From Theorem 4.4, the parametrization of all internally stabilizing controllers $C(s)$ for $G(s)$ is same to that of all internally stabilizing controllers $\bar{C}(s)$ for $\bar{G}(s)=G(s)+K(s)$. The parametrization of all internally stabilizing controllers $\bar{C}(s)$ for $\bar{G}(s)=G(s)+K(s)$ is given by (13), where $\bar{Q}(s)$ is any biproper minimum phase rational function. Equation (13) corresponds to (12). From Theorem 4.4, using $\bar{C}(s)$, $C(s)$ is written by (21). Equation (21) corresponds to (11).

This completes the proof of Theorem 4.1.

Note 4.1. *From Theorem 4.1, when $G(s)$ is strictly proper and of minimum phase, we can obtain the parametrization of all proper internally stabilizing controllers by identifying $(G(s)+K(s))^{-1}$ under preliminarily finding biproper stable function $K(s)$ to make $G(s)+K(s)$ of minimum phase.*

5. **Control Structure for Minimum Phase Systems.** From Theorem 4.1, control structure for multiple-input/multiple-output minimum phase systems is written in Figure 1. Figure 1 shows that the structure of the parametrization of all stabilizing controllers for

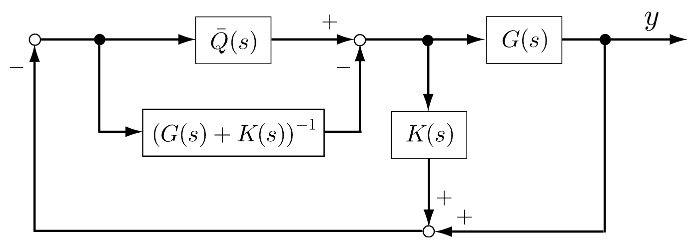

FIGURE 1. Control structure for minimum phase systems

multiple-input/multiple-output minimum phase systems includes (1) a parallel compensation for the plant $G(s)$, (2) canceling section $(G(s)+K(s))^{-1}$ for parallel compensated plant $G(s)+K(s)$ and (3) minimum phase biproper compensator $Q^{-1}(s)$.

Note 5.1. *Note that results in this section cannot be obtained using the parametrization in* [1, 2, 3, 4, 5, 6, 7]*, since* $Q(s)$ *in* [1, 2, 3, 4, 5, 6, 7] *is proper, but* $Q(s)$ *in Theorem 4.1 is biproper.*

6. **Conclusion.** In this paper, we proposed the simple parametrization of all proper internally stabilizing controllers for multiple-input/multiple-output minimum phase plants. As for applications of the result of the present paper, such as adaptive control, descriptions will be made in separate papers.

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